CPSC 536C Student Lecture: Universal Barrier

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1 Lecture Overview and Motivation

In this lecture, we will be exploring the idea of a *universal barrier*. To motivate, consider the following LP problem

$$\min_x f(x) \ s.t. \ Ax \leq b$$

for f linear, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. We can reformulate this problem as

$$\min_x f(x) \underbrace{- \sum_{i=1}^{m} \log(b_i - A_i^\top x)}_{\phi(x)} .$$

and regard $\phi(x)$ as a barrier function. We can then utilize interior point methods to solve this problem within $O(\sqrt{\nu}\log(\epsilon^{-1}))$ Newton steps, where ν is the barrier parameter of ϕ. Since barrier parameter is affine-invariant, and $-\log(\cdot)$ has barrier parameter 1, ϕ has barrier parameter m, since barrier parameters are additive. When the number of constraints m increases, our iterative complexity increases by a factor of $\sqrt{m}$. So one naturally asks the question:

Can we do better?

And the answer is yes. We will show that for all problems happening in dimension n, we can find a barrier function whose barrier parameter is at most n. In other words, when we have much more constraints than we have dimension, i.e. $m > n$, this barrier function will incur a much faster iterative complexity than ϕ, since it would be independent of number of constraints and only dependent on the dimensionality. Furthermore, we will prove that the lower bound for such a barrier function is n, meaning that we cannot do better than the universal barrier.

In practice, this construction is too complicated to realistically run algorithms, but the existence of a universal barrier is a testimony that in principle, out theory is applicable to any convex problem.

We will make a note that for any α-self-concordant function ϕ with barrier parameter ν, $c\phi$ is $c^{-1/2}\alpha$-self-concordant with barrier parameter $c^{1/2}\nu$. Thus, we can rescale a ϕ by a constant so that ϕ is $\alpha\nu$-self-concordant and with barrier parameter 1. And we can turn the problem from finding a barrier function with barrier parameter n to finding a function that is at most n-self-concordant with barrier parameter 1, and we will employ this approach in this lecture.

2 Lower Bound

2.1 Minkowsky Function

In this section, we develop a lower bound for the self-concordance parameter of a barrier function F on a convex body $K \subseteq \mathbb{R}^n$. For a given n, we give an example of body $K \subseteq \mathbb{R}^n$ such that all self-concordant barrier functions on K are at least n-self-concordant. To do so, we first introduce the concept of *Minkowsky function*.

Definition 2.1. Let $G \subseteq \mathbb{R}^n$ be a convex body with nonempty interior, for $x, y \in \text{int}(G)$, the Minkowsky function $\pi_y(x)$ is given by

$$\pi_y(x) = \inf \left\{ t \geq 0 : y + \frac{x-y}{t} \in G \right\}.$$

Intuitively, the Minkowsky function $\pi_y(x)$ measures the ratio given by the distance between x and y, divided by the largest distance traveled from point y in direction x without exiting G, as illustrated below.

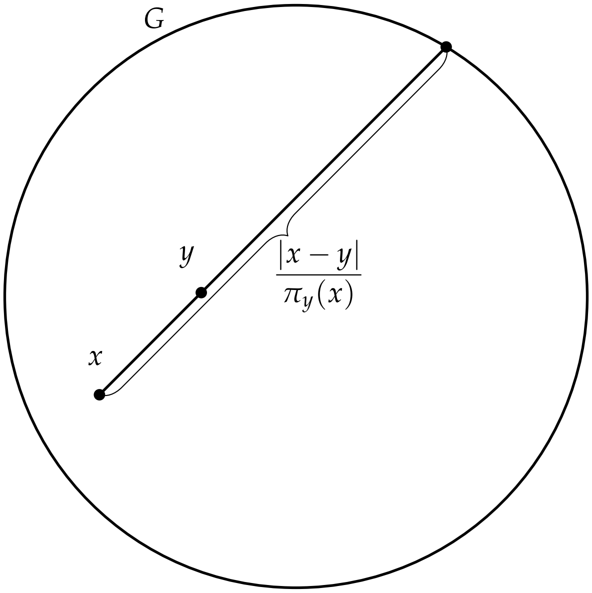

Figure 1: Illustration of the Minkowsky function on a circle

2.2 A Key Lemma

Next, we will establish two key lemmas that will help with our construction.

Lemma 2.1 (Nesterov and Nemirovskii [1994], prop. 2.3.2, 2.3.7)**.** Suppose $G \subseteq \mathbb{R}^n$ is a convex body with nonempty interior, $x, y \in \text{int}(G)$, and $F : G \to \mathbb{R}$ a α-self-concordant barrier function, then for $z \in \partial G$ such that

$$\pi_z(x) \leq (\sqrt{\alpha} + 1)^{-2},$$

we have

$$\begin{aligned} &(i) \quad DF(x)[y-x] \leq \alpha, \text{ and} \\ &(ii) \quad DF(x)[z-x] \geq 1 - (\sqrt{\alpha}+1)^2 \pi_z(x). \end{aligned}$$

Proof. We will follow the proof outlined in Nesterov and Nemirovskii [1994]. We first prove assertion (i). Let us define

$$\begin{aligned} \Delta &= \{t \in \mathbb{R} : y + t(x-y) \in \text{int}(G)\} \\ -T' &= \inf \Delta \\ T &= \sup \Delta \\ \psi(t) &= F(y + t(x-y)). \end{aligned}$$

Note that if ψ is a constant function, then $DF(x)[y-x] = 0$ and the assertion is trivial. So we assume that ψ is not a constant function. Since ψ is a barrier function on $\overline{\Delta}$, the topological closure of Δ, we have $\psi''(t) > 0$ and $\psi'(t)^2 \leq \alpha\psi''(t)$ for $t \in \Delta$, then by calculus, we have we can choose a $t_0 \in \Delta$ such that $\psi'(t_0) > 0$, we have

$$\begin{aligned} \psi'(t) &\geq \frac{\alpha\psi'(t_0)}{\alpha - (t-t_0)\psi'(t_0)}, \text{ and thus} \\ \psi'(t_0) &\leq \frac{\alpha}{T - t_0} \end{aligned}$$

and so for $0 < t < T < \infty$,

$$\psi'(0) \leq \psi'(t) \leq \frac{\alpha}{T}\frac{T}{T-t} = \alpha\pi_y(x)\frac{T}{T-t} \leq \alpha,$$

where the leftmost inequality comes from convexity of ψ. Pushing $T \to \infty$, one obtains $\psi'(0) = 0 \leq \alpha$. Thus, we have $DF(x)[y-x] \leq \alpha$ and assertion (i) is proved. Next we move on to (ii). Similar to the previous assertion, let

$$\begin{aligned} \Omega &= \{t \in \mathbb{R} : z + t(x-z) \in \text{int}(G)\} \\ -R' &= \inf \Omega \\ R &= \sup \Omega \\ \varphi(t) &= F(z + t(x-z)). \end{aligned}$$

then by $z \in \partial G$, we have $R' = 0$ and $R = \pi_z^{-1}(x) \geq (1+\sqrt{\alpha})^2$. Using the fact that φ is a barrier on the topological closure of Ω, we have $\varphi''(t) \geq t^{-2}$, so by similar argument,

$$\begin{aligned} t\varphi'(t) &\leq \frac{\alpha t}{T-t}, \text{ and} \\ \varphi'(1) + \int_i^t s^{-2}\, ds &\leq \frac{\alpha}{T-t}, \text{ and} \\ \varphi'(1) &\leq \frac{\alpha}{T-t} - 1 + \frac{1}{t} \end{aligned}$$

for $1 \leq t \leq T < \infty$. Setting $t = (1+\sqrt{\alpha})^{-1}\pi_z^{-1}(x)$ we have

$$\varphi'(1) \leq -1 + (1+\sqrt{\alpha})^2 \pi_z(x),$$

and is unchanged when pushing $T \to \infty$. Thus,

$$DF(x)[z-x] = -\varphi'(1) \geq 1 - (\sqrt{\alpha}+1)^2 \pi_z(x),$$

and the proof is complete. ■

2.3 Construction

Now we are ready to construct such K.

Theorem 2.1. For all $n \geq 1$, there exists a convex body $K \subseteq \mathbb{R}^n$ such that all barrier functions $F : K \to \mathbb{R}$, that is α-self-concordant, we have $\alpha \geq n$.

Proof. Fix n. Let we pick $K = \mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1, x_2, \ldots, x_n \geq 0\}$ to be the positive orthant and we claim that all self-concordant barrier functions on K is at least n-self concordant. For $t > 0$, let us define the function

$$\begin{aligned} x(t) &= t\sum_{i=1}^{n} e_i, \text{ and} \\ x^j(t) &= t\sum_{i \neq j} e_i \end{aligned}$$

where e_i is the i-th Euclidean basis. Intuitively, $x(t) = t\vec{\mathbf{1}}$ is the vector comprised of all 1's stretched by the scalar t, and $x^j(t)$ is $x(t)$ projected to onto the face

$$F_j := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_j = 0, x_i \geq 0 \ \forall i \neq j\} \subseteq \partial K.$$

One can intuitively verify that

$$\pi_{x^j(t)}(x(t)) \to 0 \qquad \text{as} \qquad t \to 0$$

for all polytope P such that F_j is a face of P. This stronger assertion is not needed in the case of $K = \mathbb{R}^n_+$, because, in fact $\pi_{x^j(t)}(x(t)) = 0$ for all $t > 0$ due to unboundedness of K. Now by Lemma 2.1 (ii), for all α-self-concordant barrier function ϕ, we have

$$\liminf_{t \to 0} DF(x(t))[x^j(t) - x(t)] \geq 1,$$

yet by Lemma 2.1 (i),

$$\begin{aligned} \alpha &\geq \liminf_{t \to 0} DF(x(t))[t - x(t)] \\ &= \liminf_{t \to 0} \sum_{j=1}^{n} DF(x(t))[x^j(t) - x(t)] \\ &\geq n. \end{aligned}$$

So we deduced that $a \geq n$. ■

A small remark regarding this proof is that the limit infimum is needed because Lemma 2.1 requires that x, y to be in the interior of K, in order for the bound $DF(x(t))[x-y] \leq \alpha$ to hold, thus we cannot directly assert that $\alpha \geq DF(x(t))[0 - x(t)]$. As a final note, although this construction of K is sufficient to prove the lower bound of the self-concordance parameter, it is a special case of a more general theorem given below.

Theorem 2.2 (Nesterov and Nemirovskii [1994], Proposition 2.3.6)**.** Let K be a convex polytope in $\mathbb{R}^n$ such that a certain boundary point $p \in \partial K$ belongs exactly to k of $(n-1)$ dimensional facets of K, with the normals to these facets being linearly independent. Then the value of the parameter of any α-self-concordant function F must satisfy $\alpha \geq k$. In particular, the n-dimensional nonegative orthant, simplex, and cube do not admit barriers with parameter less than n.

The proof of Theorem 2.2 is almost identical to the proof of Theorem 2.1, the generalization being that Theorem 2.2 considers general facets instead of $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_j = 0, x_i \geq 0\ \forall i \neq j\}$. In this case, the argument $\pi_{x^{j}(t)}(x(t)) \to 0$ is needed since we no longer have $\pi_{x^{j}(t)}(x(t)) = 0$ for all t in the case of unbounded polytope.

3 Upper Bound

3.1 Universal Barrier

In this section we will present the construction of a *universal barrier*, that is, given any polytope $K \subseteq \mathbb{R}^n$, we construct in an universal way a α-self-concordant barrier function such that $\alpha \leq O(n)$. The construction is as follows: given any polytope K, we will claim that

$$F(x) = \log(|K^*(x)|)$$

is a universal barrier, where $K^*(x)$ is the polar set, given by

$$K^*(x) = \{y \in \mathbb{R}^n : y^\top (z - x) \leq 1\ \forall z \in K\},$$

and $|\cdot|$ is the n-dimensional Lebesgue measure.

3.2 A Basic Example

The construction of the universal barrier is quite abstract and in this section we aim to give a concrete example of K, $K^*(x)$, and F. In particular, we present visually how the set $K^*(x)$ changes when x is pushed towards the boundary of K, and why F will diverge to ∞ when this happens.

Let K be the unit circle in $\mathbb{R}^2$ centered around the origin O, then the polar set $K^*(x)$, up to a translation, is given by the polar equation $r(\theta) \leq (1 + \rho\cos(\theta))^{-1}$. This equation defines a conic section with eccentricity ρ. When $0 < \rho < 1$, the resulting conic section is an ellipse whose area is given by $\pi\frac{1-\rho+\rho^2}{(1-\rho)^{3/2}}$, and the universal barrier function is given by $F(\rho) = \log(\pi) + \log(1-\rho+\rho^2) - \frac{3}{2}\log(1-\rho)$.

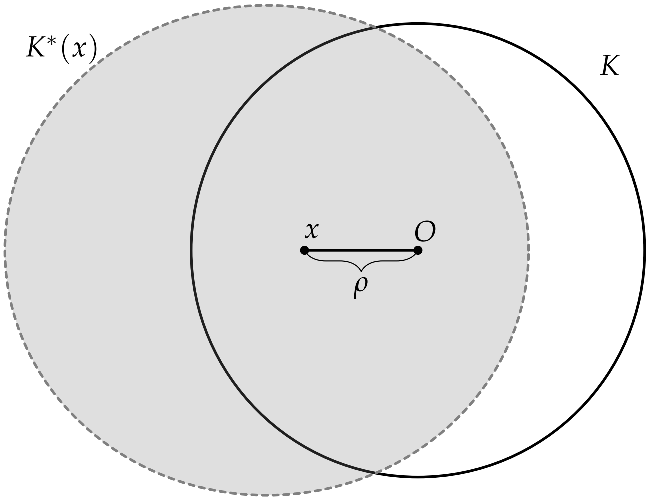

Figure 2: $K^*(x)$ when x is close to the origin.

When we push x into the boundary of K, the set $K^*(x)$ becomes a conic section with eccentricity 1, which is a parabola. In this case, its area is infinity, and this F diverges to infinity.

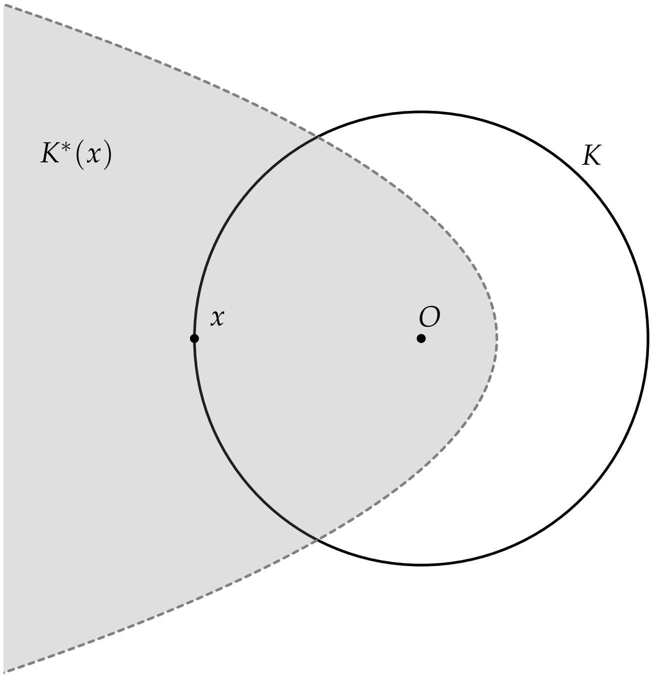

Figure 3: $K^*(x)$ when x is pushed into ∂K.

3.3 Key Tools to Establishing Upper Bound

We will follow the proof by Lee and Yue [2021], which is a recent improvement to the known result established Nesterov and Nemirovskii [1994]. Compared to Lee and Yue [2021]'s sharp bound of $\alpha \geq n$, Nesterov's known result only asserts that the self-concordance function of the universal barrier is $O(n)$.

Before beginning the proof, we will motivate a few concepts that will serve as key ingredients in the proof. In the process of establishing n-self-concordance of the universal barrier, the evaluation of the volume $|K^*(x)|$ is unavoidable. The strategy that simplifies this process is rewriting the volume, which is an integral over $K^*(x)$, as an expectation of a suitable random variable Y. The higher order derivatives of the barrier function can thus in turn be rewritten using higher moments of Y, and can be studied using by studying the probability mass function of Y. We will introduce the properties and theorems of interest below.

Definition 3.1 (m-concavity)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function, we say that f is m-concave if for all $a, b \in \mathbb{R}$, and $\lambda \in [0,1]$, there exists parameter m independent of a, b and λ such that

$$f(\lambda a + (1-\lambda)b)^{1/m} \geq \lambda f(a)^{1/m} + (1-\lambda) f(b)^{1/m}.$$

Theorem 3.1 (Adapted from Proposition 1 of Lee and Yue [2021])**.** Let Y be a random variable with mean μ, and let $p_Y(t)$ denote its probability mass function. If $p_Y(t)$ is $\frac{1}{n-1}$-concave, then we have

$$\int_{-\infty}^{\infty} (t-\mu)^3 p_Y(t)\, dt \leq 2\sqrt{\frac{n+2}{n}\frac{n-1}{n+3}} \left(\int_{-\infty}^{\infty} (t-\mu)^2 p_Y(t)\, dt \right)^{3/2},$$

or equivalently, we have $\theta \leq 2\sqrt{\frac{n+2}{n}\frac{n-1}{n+3}}\sigma^3$, under the notation

$$\sigma^2 = \mathbb{E}[(Y-\mu)^2] \qquad \text{and} \qquad \theta = \mathbb{E}[(Y-\mu)^3].$$

Related to this concept is the notion of log-concavity. This property is unused in our proof, but rather it is the property used by Nesterov and Nemirovskii [1994] to establish the looser bound. The utilization of m-concavity in place log-concavity is the deviation of Lee and Yue [2021]'s proof from Nesterov's proof, and to a large extent contributed to the sharpened bound.

Definition 3.2 (log-concavity)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function, we say that f is log-concave if for all $a, b \in \mathbb{R}$, and $\lambda \in [0,1]$, we have

$$f(\lambda a + (1-\lambda)b) \geq f(a)^{\lambda} f(b)^{1-\lambda}.$$

Remark 3.1. One should remark that m-concavity is a stronger condition than log-concavity. From Young's inequality we have

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

for $\frac{1}{p} + \frac{1}{q} = 1, u, v \geq 0$. Then from m-concavity we have that for all a, b

$$f(\lambda a + (1-\lambda)b)^m \geq \lambda f(a)^m + (1-\lambda)f(b)^m.$$

Choosing $u = f(a)^m, v = f(b)^m, p = 1/\lambda, q = 1/(1-\lambda)$ we have

$$\lambda f(a)^m + (1-\lambda)f(b)^m \geq f(a)^{m\lambda} f(b)^{m(1-\lambda)}$$

and so

$$f(\lambda a + (1-\lambda)b) \geq f(a)^\lambda f(b)^{1-\lambda}.$$

3.4 Proof of Upper Bound

We will hereby formulate and prove the upper bound statement of barrier functions on polytopes.

Theorem 3.2. Let K be an n-dimensional polytope, then there exists a barrier function $F : K \to \mathbb{R}$ such that F is n-self concordant with barrier parameter 1. More precisely, we can construct F to be

$$F(x) = \log(|K^*(x)|),$$

where $K^*(x)$ is the polar set. We call F the *universal barrier*.

proof of Theorem 3.2. Recall for a n-self concordant function f with barrier parameter 1, the function $n^{1/2}f$ is 1-self concordant with barrier parameter n. Thus, we can assume that F has barrier parameter $\nu = n$ without the loss of generality, and our proof reduces to proving 1-self concordance of F.

Let $h \in \mathbb{R}^n$ be an arbitrary vector and let $t \in \mathbb{R}$ be a small scalar. We consider $K^*(x)$ under the small perturbation $x \to x + th$:

$$\begin{aligned} K^*(x+th) &= \{y \in \mathbb{R}^n : y^\top (z - x - th) \leq 1 \ \forall z \in K\} \\ &= \{y \in \mathbb{R}^n : y^\top z \leq 1 + ty^\top h \ \forall z \in K - x\} \\ &= \left\{ y \in \mathbb{R}^n : \frac{y^\top z}{1 + ty^\top h} \leq 1 \ \forall z \in K - x \right\} \\ &= \left\{ \frac{u}{1 - tu^\top h} : u^\top z \leq 1 \ \forall z \in K - x \right\} \end{aligned}$$

using the substitution

$$u = \frac{y}{1 + ty^\top h}, \qquad \text{and thus} \qquad y = \frac{u}{1 - tu^\top h}.$$

So we have

$$|K^*(x+th)| = \int_{K^*(x+th)} dy = \int_{K^*(x)} \det\left(\frac{\partial y}{\partial u}\right) du.$$

From the definition of u, we have

$$\frac{\partial y}{\partial u} = \underbrace{\frac{1}{1 - tu^\top h}}_{a} I_n - \underbrace{\frac{tu^\top h}{(1 - tu^\top h)^2}}_{b} uh^\top$$

which in turn can be written as $aT_n + buh^\top$, which is the sum of a scaled identity matrix and a rank 1 outer product. Thus, the eigenvalues of this matrix are $a + b^\top u$ with multiplicity 1 and a with multiplicity $n - 1$. The determinant is then easily computable: $\det(aT_n + buh^\top) = a^{n-1}(a + bh^\top u)$, and so

$$\begin{aligned} \det\left(\frac{\partial y}{\partial u}\right) &= \left(\frac{1}{1 - tu^\top h}\right)^{n-1} \left(\frac{1}{1 - tu^\top h} + \frac{tu^\top h}{(1 - tu^\top h)^2}\right) \\ &= \frac{1}{(1 - tu^\top h)^{n+1}}. \end{aligned}$$

Next, we will simplify the integral by interpreting it as scaled expectation

$$\begin{aligned} \int_{K^*(x)} \det\left(\frac{\partial y}{\partial u}\right) du &= \int_{K^*(x)} \frac{1}{(1 - tu^\top h)^{n+1}}\, du \\ &= |K^*(x)| \mathbb{E}[(1 - tY)^{-n+1}], \end{aligned}$$

where Y is the random variable given by $Y = u^\top h$, $u \sim \mathrm{Unif}(K^*(x))$. Now, we can define

$$M(t) = \mathbb{E}[(1 - tY)^{-(n+1)}]$$

so that

$$\begin{aligned} M'(0) &= (n+1)\mathbb{E}[Y] \\ M''(0) &= (n+2)(n+1)\mathbb{E}[Y^2] \\ M'''(0) &= (n+3)(n+2)(n+1)\mathbb{E}[Y^3]. \end{aligned}$$

Going back to the universal barrier F, we have

$$\begin{aligned} F(x + th) &= \log(|K^*(x + th)| \\ &= \log(|K^*(x)|) + \log \mathbb{E}[(1 - tY)^{-(n+1)}] \\ &= \log(|K^*(x)|) + \log M(t) \end{aligned}$$

and by the chain rule,

$$\begin{aligned} DF(x)[h] &= (n+1)\mathbb{E}[Y] \\ D^2F(x)[h,h] &= M''(0) - M(0)^2 \\ &= (n+2)(n+1)\mathbb{E}[Y^2] - (n+1)^2\mathbb{E}[Y]^2 \\ D^3F(x)[h,h,h] &= M'''(0) - 3M''(0)M'(0) + 2M'(0)^3 \\ &= (n+3)(n+2)(n+1)\mathbb{E}[Y^3] - 3(n+2)(n+1)^2\mathbb{E}[Y^2]\mathbb{E}[Y] \\ &\quad + 2(n+1)^3\mathbb{E}[Y]^3. \end{aligned}$$

Writing

$$\mu = \mathbb{E}[Y], \qquad \sigma^2 = \mathbb{E}[(Y-\mu)^2], \text{ and } \qquad \theta = \mathbb{E}[(Y-\mu)^3],$$

we have

$$\begin{aligned} D^2F(x)[h,h] &= (n+2)(n+1)\sigma^2 + (n+1)\mu^2 && (1) \\ D^3F(x)[h,h,h] &= (n+3)(n+2)(n+1)\theta + 6(n+2)(n+1)\sigma^2\mu + 2(n+1)\mu^3. && (2) \end{aligned}$$

Recall the barrier parameter is given by

$$\begin{aligned} \nu &= \sup_{x\in K} \left\| \nabla F(x)^2 \right\|_{x,*} \\ &= \sup_{x\in K} \sqrt{\frac{(DF(x)[h])^2}{DF(x)[h,h]}} \\ &= \sqrt{\frac{(n+1)^2\mu^2}{(n+2)(n+1)\sigma^2}} \\ &= n. \end{aligned}$$

This establishes

$$\sigma \le \frac{\sqrt{n+1}}{n\sqrt{n+2}}\mu,$$

and if moreover we have

$$\theta \le 2\sqrt{\frac{n+2}{n}}\frac{n-1}{n+3}\sigma^3,$$

then using Eq. (1), (2), we can express $p(n) = (D^3F(x)[h,h,h])^2 - 4(D^2F(x)[h,h])^3$ as a polynomial solely in n and parameter μ. By voluminous direct computation, which we will refer the readers to Lee and Yue [2021] Section 3 Eq. (8) where this computation is performed, we shall have $p(n) \le 0$ for all $n \ge 0$, and thus the following desired result:

$$D^3F(x)[h,h,h] \le 2(D^2F(x)[h,h])^{3/2}.$$

It remains for us to establish the all-important bound

$$\theta \le 2\sqrt{\frac{n+2}{n}}\frac{n-1}{n+3}\sigma^3.$$

By Theorem 3.1, we can establish this by proving $\frac{1}{n-1}$-concavity of the probability mass function of the random variable Y. Let $p_Y(t)$ denote the mass function of Y, then $p_Y(t)$ is given by

$$p_Y(t) = \frac{\mathrm{Vol}_{n-1}(\{y\in K : y^\top h = t\})}{\mathrm{Vol}_n(K)},$$

where $\mathrm{Vol}_n(\cdot)$ and $\mathrm{Vol}_{n-1}(\cdot)$ denote volume in n and in $n-1$ dimensions (with respect to Lebesgue measure). We will establish log-concavity and $\frac{1}{n-1}$-concavity of $p_Y(t)$. We make the substitution $y = tz$ and rewrite the set $\{y \in K : y^\top h = t\}$ as

$$\begin{aligned}\{y \in K : y^\top h = t\} &= \{tz \in K : z^\top h = 1\}\\ &= t\{z \in K : z^\top h = 1\},\end{aligned}$$

so we have that $p_Y(t) = t^{n-1} p_Y(1)$, since $\{y \in K : y^\top h = t\}$ is an $(n-1)$-dimensional object. Then it follows very naturally from this property that for $a, b \in \mathbb{R}$, $\lambda \in [0, 1]$

$$\begin{aligned}p_Y(\lambda a + (1-\lambda) b)^{\frac{1}{n-1}} &= (\lambda a + (1-\lambda) b) p_Y(1)^{\frac{1}{n-1}}\\ &= \lambda a p_Y(1)^{\frac{1}{n-1}} + (1-\lambda) b p_Y(1)^{\frac{1}{n-1}}\\ &= \lambda p_Y(a)^{\frac{1}{n-1}} + (1-\lambda) p_Y(b)^{\frac{1}{n-1}},\end{aligned}$$

which establishes $\frac{1}{n-1}$-concavity of $p_Y(t)$. This final statement establishes our universal barrier. ■

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