Continuous Optimization: Problem Set 3

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1. Recall the definition of α-self-concordance and barrier parameter θ:

$$\forall u, v, w : |D^3\phi(x)[u, v, w]| \leq \frac{2}{\sqrt{\alpha}}\|u\|_x\|v\|_x\|w\|_x, \qquad \theta := \sup_{x \in D} \|(\nabla^2\phi(x))^{-1}\nabla\phi(x)\|_x.$$

   1. Assume ϕ_k are each α_k-self-concordant with barrier parameter θ_k for domain D. Compute the self-concordance and barrier parameter for $\sum_k \phi_k$.
   2. Show $-\log\det(X)$ is self-concordant for $D := \{X \succ 0\}$, and compute its barrier parameter.

2. (Theorem 4.1.3 in Nesterov) We made the assumption that $\nabla^2 \succ 0$ everywhere. In this question we remove the assumption. Show if there is some direction such that $\langle v, \nabla^2\phi(x)v\rangle = 0$ and ϕ is *self-concordant*, then in fact

$$\forall y \in \mathrm{dom}(\phi) : \langle v, \nabla^2\phi(y)v\rangle = 0.$$

   What does this say about the domain in this direction?

3.
   1. Prove affine invariance of the Newton step: for convex f let $g(y) := f(Ay)$ so for $y = A^{-1}x$ $g(y) = f(x)$. Compute the gradient and Hessian and Newton step and show $y_{t+1} = A^{-1}x_{t+1}$ if $y_t = A^{-1}x_t$.
   2. Prove affine invariance of the definition of self-concordance and barrier parameter.

4. (Exercise 9.10 in BV): Newton's method with fixed step size $\lambda = 1$ can diverge if the initial point is not close to x^*. In this problem we consider two examples.

   a. $f(x) = \log(e^x + e^{-x})$ has unique minimizer $x^* = 0$. Run Newton's method with fixed step size $t = 1$, starting at $x^0 = 1$ and at $x^0 = 1.1$.

   b. $f(x) = -\log(x) + x$ has unique minimizer $x^* = 1$. Run Newton's method with fixed step size $\lambda = 1$, starting at $x^0 = 3$.

5. (Daniel Dadush Course, Ex 9.11) Let $\mathsf{A} \in \mathbb{R}^{m\times n}$, $\mathrm{rank}(\mathsf{A}) = m$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$. For $\mu > 0$, examine the optimization problems

$$\min_{\mathbf{x}}\{\mathbf{c}^\mathsf{T}\mathbf{x} - \mu\sum_{i=1}^n \ln x_i : \mathsf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} > 0\} \tag{BP}$$

$$\max_{\mathbf{s},\mathbf{y}}\{\mathbf{b}^\mathsf{T}\mathbf{y} + \mu\sum_{i=1}^n \ln s_i : \mathsf{A}^\mathsf{T}\mathbf{y} + \mathbf{s} = \mathbf{c}, \mathbf{s} > 0\}. \tag{BD}$$

Assume that the feasible regions for both programs are non-empty (i.e., the primal and dual LPs are both strictly feasible).

1. Show that the Lagrangian dual of BP is equivalent to BD and prove that strong duality holds.
2. Conclude that $\mathbf{x}_\mu$ is an optimal to solution to (BP) iff $\exists$ a solution $\mathbf{s}_\mu, \mathbf{y}_\mu$ to (BD) satisfying $x_{\mu,i}s_{\mu,i} = \mu$, $\forall i \in [n]$.